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ON ERGODIC QUASI-INVARIANT MEASURES ON THE CIRCLE GROUP*

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I. Introduction. In their generalization of F. and M. Riesz' Theorem to compact abelian groups with ordered duals [1], K. deLeeuw and I. Glicksberg introduced analytic measures and showed that they are quasi-invariant under a subgroup (see also [6]). In [3] F. Forelli generalized the results of [1] and established that the extreme points of analytic measures are ergodic. Thus in order to obtain the structure of analytic measures one has to know the class of quasi-invariant ergodic measures. In this paper we characterize by suitable conditions a class of ergodic measures on the circle quasi-invariant under a dense group. We show that the conditions of this theorem are not necessary by exhibiting an example of an ergodic quasi-invariant measure not satisfying our condition. Finally we remark that it is easy to translate our results into results on ergodic measures on the real line quasi-invariant under a dense subgroup. Our results also answer certain questions raised in [7].

II. Let T denote the circle group with the usual topology under which it is a compact abelian group. We shall denote the group operation by $+$. Let G be a fixed countable dense subgroup of T. For any two measures μ and ν on the Borel subsets $\mathcal{B}$ of T, we shall write $\mu \equiv \nu$ when μ and ν are mutually absolutely continuous. We shall also write μ_g to denote the measure defined by $\mu_g(A) = \mu(A + g)$, $A \in \mathcal{B}$, $g \in G$. A set $A \subset T$ is called G-invariant if $A + g = A$.

Definition 1: A finite measure μ on $\mathcal{B}$ is called G-quasi-invariant if $\mu \equiv \mu_g$ for all $g \in G$.

Definition 2: A G-quasi-invariant measure μ is called ergodic if for any measurable G-invariant set A, either $\mu(A) = 0$ or $\mu(T - A) = 0$.

Any measure μ mutually absolutely continuous with respect to the Haar measure on T is an example of an ergodic measure. Also, any measure mutually absolutely continuous with respect to the cardinality measure on a coset of G is an ergodic measure. In this section we state some elementary properties of ergodic measures without proof. Next we introduce a subset of T and in terms of this set and the given measure to be one of the above type.

<u>Theorem 1</u>: Let μ and ν be two ergodic measures on T. Then

(a) if A is a set with positive outer μ measure such that $\bigcup_{g \in G} (A + g)$ is measurable, then $\bigcup_{g \in G} (A + g)$ supports μ;

(b) either $\mu \equiv \nu$ or μ and ν are mutually singular;

(c) $\mu * \nu$ is ergodic, where $*$ denotes the convolution.

Proof of this theorem, being easy, is omitted.

For any measure μ we shall write $F_\mu = \{x: \mu_x \equiv \mu, x \in T\}$. It is easy to verify that F_μ is a group. We would like to show that F_μ is a measurable set. Proof of this fact relies on a theorem of Doob ([2], p. 616) which we state here in a form convenient to us.

<u>Theorem 2</u>: Let p be a finite measure on $\mathcal{B}$. Let $\mu(\cdot,\cdot)$ be a function on $\mathcal{B} \times T$ such that for each $t \in T$, $\mu(\cdot,t)$ is a finite measure on $\mathcal{B}$ and for each $E \in \mathcal{B}$, $\mu(E,\cdot)$ is a measurable function. Then there exists a jointly measurable function $f(\cdot,\cdot)$ on $T \times T$ such that

$$\mu(E,t) = \int_E f(w,t)\, p(dw) + \nu(E,t)$$

where $\nu(\cdot,t)$ is a measure singular with respect to p.

<u>Corollary 1</u>: In Theorem 2 $\{t: \nu(T,t) = 0\}$ is a measurable set.

<u>Proof</u>: $\nu(T,t) = \mu(T,t) - \int_T f(w,t)p(dw)$. Since $f(\cdot,\cdot)$ is a jointly measurable function $\int_T f(w,\cdot)p(dw)$ is a measurable function. Also $\mu(T,\cdot)$ is a measurable function so that $\{t: \nu(T,t) = 0\}$ is a measurable set.

q.e.d.

<u>Theorem 3</u>: F_μ is a measurable set.

<u>Proof</u>:* In Theorem 2, take $\mu(\cdot,\cdot)$ defined by $\mu(E,t) = \mu_t(E) = \mu(E + t)$ and take $p = \mu$. It is known that this function is measurable in t and for each fixed t it is easy to see that $\mu(\cdot,t)$ is a measure on $\mathcal{B}$. Hence by Theorem 2

$$\mu_t(E) = \int_E f(w,t)\,\mu(dw) + \nu(E,t).$$

Now $\mu_t \equiv \mu$ if and only if $\nu(T,t) = 0$ so that $F_\mu = \{t: \mu_t \equiv \mu\} = \{t: \nu(T,t) = 0\}$ is a measurable set by Corollary 1.

q.e.d.

Now suppose that μ is G-quasi-invariant. Then $G \subsetneq F_\mu$ and μ is F_μ-quasi-invariant. If μ is ergodic with respect to G, then it is obviously ergodic with respect to F_μ.

<u>Main Theorem</u>: Let μ be a G-quasi-invariant measure which is ergodic. Then $\mu(F_\mu + x) > 0$ for some $x \in T$ is and only if μ is mutually absolutely continuous with respect to either the Haar measure on T or the cardinality measure on $G + x$.

Proof of this theorem relies on a theorem of Mackey ([5], p. 146) which we state here in a less general form convenient to us.

<u>Theorem 4</u>: Let (X,S,m) be a measure space such that

(i) X is an abelian group;

(ii) $\{x\} \in S$, where $x \in X$;

(iii) S is an σ-algebra which is countably generated and which is invariant under translation by members of X, i.e., $A \in S$ implies $A + x \in S$ for all $x \in X$;

(iv) $m_x \equiv m$ for all $x \in X$, where $m_x(A) = m(A + x)$, $A \in S$;

(v) there exists a measurable function from a Borel subset of the real line onto X.

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*Doob's result and the proof of Theorem 3 was shown to us by Professor B. Jamison.

Then there exists a topology J on X such that

(a) (X,J) is a separable locally compact topological group;

(b) $S = \sigma$-algebra generated by J.

<u>Proof of the Main Theorem</u>: The necessity part is easy to see; we prove the sufficiency. Suppose that $\mu(F_\mu + x) > 0$ for some x. Without loss of generality we can assume that $\mu(F_\mu) > 0$ (for otherwise we can look at the measure μ_x). Now consider $(F_\mu, F_\mu \cap \mathcal{B}, \mu)$. All the conditions of Theorem 4 are satisfied by $(F_\mu, F_\mu \cap \mathcal{B}, \mu)$. Hence there exists a topology J on F_μ such that (F_μ,J) is a locally compact topological group and the σ-algebra generated by $J = F_\mu \cap \mathcal{B}$. Now from the structure of locally compact abelian groups we know that there exists an open subgroup $H \subset F_\mu$ such that H is the direct sum of a compact group K and a euclidean space R^n $(n \geq 0)$ ([8], p. 40). First we show that $n = 0$. For suppose $n > 0$. Then there exists a homeomorphic isomorphism φ of R, the real line with the usual topology, into F_μ. $\varphi(R)$ is a measurable subgroup of F_μ and hence of T. Therefore φ is a measurable (hence continuous) homomorphism of R into T which is one-one, which is impossible. Hence $n = 0$.

Now consider K. If K is a finite group, then, since K is open, J must be discrete topology on F_μ. Since (F_μ,J) is separable, F_μ is countable. Since μ is ergodic, $F_\mu = G$. (For if $F_\mu \neq G$, then there exists a coset of G in F_μ distinct from G. This coset is G-invariant with positive μ measure, contradicting the ergodicity of μ.) Suppose now that K is infinite. Then K must be dense in T. Further K is a measurable subgroup of T. The map $\varphi: K \to T$ given by $\varphi(x) = x$ is a measurable (hence continuous) isomorphism of K into T. Since K is compact, $\varphi(K) = K$ is a compact subset of T which is dense in T. Hence $K = T$.

q.e.d.

We can deduce from this theorem the following corollary on quasi-invariant ergodic measures on R.

<u>Corollary</u>: Let μ be a finite measure defined on the Borel subsets of R such that it is quasi-invariant and ergodic under translation by a countable dense subgroup of $G \subset R$. Then $\mu(F_\mu + x) > 0$ for some $x \in G$ if and only if either $\mu \equiv$ cardinality measure on a coset of G or $\mu \equiv$ Lebesgue measure on R.

<u>Proof</u>: Let $g > 0$, $g \in G$. We regard the closed interval $[0,g]$ as a group of real numbers mod g. Let $\bar{G}$ be the subgroup of $[0,g]$ consisting of $[0,g] \cap G$, i.e., $\bar{G}$ is G (mod g). Since μ is quasi-invariant and ergodic under G, it can be seen that $\bar{\mu} = \mu$ restricted to $[0,g]$ is quasi-invariant and ergodic under $\bar{G}$. Further if $\mu(F_\mu + x) > 0$ for some $x \in R$, then $\bar{\mu}(F_{\bar{\mu}} + \bar{x}) > 0$ where $\bar{x} \in [0,g]$, $\bar{x} = x$ (mod g). Now we can apply the main theorem.

It follows from the method of this theorem that if H is an uncountable measurable subgroup of T, then H cannot support a finite measure which is quasi-invariant under translation by members of H unless $H = T$. We have stated our result for the circle. But a similar result for a two dimensional torus is: "Let μ be a finite measure on T^2 which quasi-invariant and ergodic under translation by a countable dense subgroup G of T^2. Then $\mu(F_\mu + x) > 0$ for some $x \in T^2$ if and only if either (i) $\mu \equiv$ cardinality measure on G; (ii) $\mu \equiv$ linear measure on one-one continuous imbedding of R into T^2; or (iii) $\mu \equiv$ Haar measure on T^2." In the case of a circle, (ii) is ruled out.

We have not yet shown that there are nonatomic ergodic measures on T singular with respect to the Haar measure on T. In the next section we exhibit one example of such a measure and obtain some elementary facts about it.

$\mu \equiv$ Lebesgue measure on R.

Proof: Let $g > 0$, $g \notin Q$. We regard the closed interval $[0,g]$ as a group of real numbers mod g. Let $\tilde{T}$ be the subgroup of $[0,g]$ consisting of $[0,g] \cap Q$, i.e., $\tilde{T} \cong Q \pmod{g}$. Since μ is quasi-invariant and ergodic under [illegible]

[illegible]

uncountable measurable subgroup of T. Then H cannot support a finite [illegible]

[illegible]

We have not yet shown that there are nonatomic ergodic measures on T singular with respect to the Haar measure on T. In the next section we exhibit one example of such a measure and obtain some elementary facts about it.

III. Let I denote the interval $0 \leq x < 1$. Expand every $x \in I$ in its ternary expansion $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$, $x_i = 0, 1, 2$. We make the expansion unique by requiring that the number of terms in the expansion be minimum. Thus if a number has two ternary expansions, we choose the one with smaller number of terms. Let $E = \{x: x_i = 0 \text{ or } 2\}$. Thus E is the well known Cantor ternary set with a small modification, which is done for convenience rather than necessity. Write for $x \in E$, $\psi(x) = \sum_{n=1}^{\infty} \frac{x_i}{2^{i+1}}$. ψ is a one-one continuous function from E onto I. Further, ψ is strictly increasing on E. Let ν be the measure on E induced by ψ; $\nu(\psi^{-1}(A)) = L(A)$, where L stands for the Lebesgue measure on I and A is any Borel subset of I. Let G be the set of real numbers in I having finitely many terms in their ternary expansion and regard G as a group of real numbers mod 1. Let $g_1, g_2, g_3, \ldots$ be a denumeration of G. Write μ for the measure defined by $\mu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \nu(A + g_n)$, where the addition $A + g_n$ is done modulo 1. Clearly μ is a finite measure on I quasi-invariant under G and μ is nonatomic and singular with respect to the Lebesgue measure on I. (We are regarding I as a group of real numbers mod 1. We know that this group is isomorphic to the circle group and that the Lebesgue measure on I is the Haar measure.)

Theorem 5: μ is ergodic, i.e., if A is a measurable subset of I such that $A + g = A$ (mod 1), for all $g \in G$, then either $\mu(A) = 0$, or $\mu(I - A) = 0$.

Proof: Suppose μ is not ergodic. Then there exists a measurable set A such that $A + g = A$, (mod 1), $\mu(A) > 0$, and $\mu(I - A) > 0$. Since E supports ν, it is clear that $\bigcup_{g \in G} (E + g)$ supports μ. Let $E_1 = E \cap A$. We show that $\psi(E_1)$ is invariant under translation by the group H consisting of real numbers (mod 1) having finitely many terms in their binary expansions.

(Here also we make the expansion unique by requiring that the number of terms in the expansion be minimum.) Let $\alpha \in \psi(E_1)$. Let $\alpha = \alpha_1\alpha_2\alpha_3,\ldots,\ldots$ be its binary expansion. Let $h \in H$ have binary expansion $h_1h_2,\ldots,h_n$. Then it is clear that $\alpha + h \pmod 1$ and α agree in their binary expansions from term $(n + 1)$ onward. Hence, from the way ψ is defined, it is clear that $\psi^{-1}(\alpha + h) - \psi^{-1}(\alpha) = g$ has finitely many terms in its ternary expansion so that $g \in G$. Now $\alpha \in \psi(E_1)$, and therefore $\psi^{-1}(\alpha) \in E_1$. Consequently $\psi^{-1}(\alpha + h) = \psi^{-1}(\alpha) + g \in E_1$. (Recall that $E_1 = E \cap A$, and A is invariant under G.) Hence $\alpha + h = \psi(\psi^{-1}(\alpha + h)) \in \psi(E_1)$. This shows that $\psi(E_1)$ is invariant under translation by members of H. Similarly we can show that $\psi(E_2)$, $E_2 = E \cap (I - A)$, is invariant under translation by members of H. Now $\psi(E_1),\psi(E_2)$ are both measurable sets with $L(\psi(E_1)),L(\psi(E_2)) > 0$. (This is because $\nu(E_1) > 0$, $\nu(E_2) > 0$.) This is impossible since the Haar measure on I (regarded as a group of real numbers and modulo 1) is ergodic under any countable dense subgroup. Hence μ is ergodic.

q.e.d.

It is an important question in ergodic theory to know whether a given quasi-invariant measure is mutually absolutely continuous with respect to a σ-finite measure invariant under the same group of transformations. Many necessary and sufficient conditions for the existence of such an equivalent invariant measure are known. However, generally, these conditions are difficult to verify. For the measure μ of this section we can show directly the existence of an equivalent σ-finite measure invariant under G. We shall prove this using the following property of the measure ν, ([4], p. 19).

(*) If A, $A + t$ are both subsets of E, then $\nu(A) = \nu(A + t)$, A being a measurable set.

Now let $D_1, D_2, D_3, \ldots$ be pairwise disjoint measurable sets such that (i) each D_i is a subset of $E + g$ for some $g \in G$, and (ii) $\bigcup_{i=1}^{\infty} D_i = \bigcup_{g \in G} (E + g)$. Define m_i on D_i by $m_i(D_i \cap A) = \nu_{-g}(D_i \cap A)$, where $D_i \subset E + g$. We extend m_i to I by making it zero on measurable sets outside D_i. Because of the property $*$ of the measure ν it is easy to check that the definition of m_i is unambiguous, i.e., if $D_i \subset E + h$, $h \neq g$, then $\nu(D_i \cap A - g) = \nu(D_i \cap A - h)$. Now write $m(A) = \sum_{i=1}^{\infty} m_i(A \cap D_i)$. Now m is an σ-finite measure since $m(D_i) < \infty$ for each i. Further, because of property $*$ of ν, m is invariant under translation by members of G. Finally m and μ are mutually absolutely continuous. For suppose $\mu(A) = 0$, then $\nu_g(A) = 0$ for every $g \in G$, so that $m(A) = 0$. Conversely, if $m(A) = 0$, then $m(A \cap (E + g)) = 0$ for every $g \in G$, so that $\nu_{-g}(A) = 0$. Hence $\mu(A) = 0$.

Remark: Although the measure m is not located on the orbit under G of a fixed point $x \in I$, there is a measure λ on I invariant under every t so that m is a restriction of λ to the set $\bigcup_{g \in G} (E + g)$. This is seen as follows. Let S be the σ-ring of measurable subsets A of I such that A is covered by countably many translates of E. We define the measure λ on S as follows: Let $A \in S$. Then $A \bigcup_{i=1}^{\infty} (E + t_i)$ for some countable set $t_1, t_2, t_3, \ldots$ in I, addition $E + t_i$ being modulo 1. Let $D_1, D_2, D_3, \ldots$ be pairwise disjoint measurable sets such that $\bigcup_{i=1}^{\infty} D_i = \bigcup_{i=1}^{\infty} (E + t_i)$, and each $D_i \subset E + t_j$ for some j. Now define $\lambda(A) = \Sigma \nu(A \cap D_i - t_j)$, where $D_i \subset E + t_j$. It can be checked that λ is a measure on S which is σ-finite in the sense that every set $A \in S$ is covered by countably many sets of finite measure. Further because of the property $*$, λ is invariant. Finally, when λ is restricted to the set $\bigcup_{g \in G} (E + g)$, we get m.

q.e.d.

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